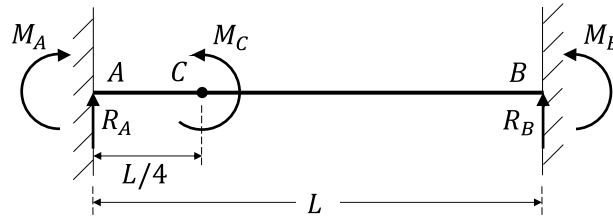


2015-2016 MM2MS3 Exam Solutions

1.

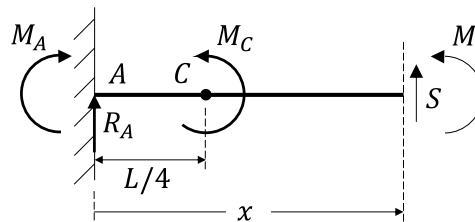
(a)

Drawing a free body diagram of the beam:



[2 marks]

Take origin from right hand side and sectioning the beam after the last discontinuity:



[1 mark]

Taking moments about the section position (remembering to use Macaulay brackets where needed) gives:

$$M + M_C \left\langle x - \frac{L}{4} \right\rangle^0 = R_A x + M_A$$

$$\therefore M = R_A x + M_A - M_C \left\langle x - \frac{L}{4} \right\rangle^0$$

[1 mark]

Substituting this in the main deflection of beams equation gives:

$$EI \frac{d^2 y}{dx^2} = M = R_A x + M_A - M_C \left\langle x - \frac{L}{4} \right\rangle^0$$

[1 mark]

Integrating gives:

$$EI \frac{dy}{dx} = \frac{R_A x^2}{2} + M_A x - M_C \left\langle x - \frac{L}{4} \right\rangle + A \quad (1)$$

[1 mark]

Integrating again gives:

$$EIy = \frac{R_A x^3}{6} + \frac{M_A x^2}{2} - \frac{M_C \langle x - \frac{L}{4} \rangle^2}{2} + Ax + B \quad (2)$$

[1 mark]

Boundary conditions:

(BC1) At $x = 0$, $\frac{dy}{dx} = 0$, therefore from (1):

$$A = 0$$

[1 mark]

(BC2) At $x = 0$, $y = 0$, therefore from (2):

$$B = 0$$

[1 mark]

(BC3) At $x = L$, $\frac{dy}{dx} = 0$, therefore from (1):

$$0 = \frac{R_A L^2}{2} + M_A L - M_C \langle L - \frac{L}{4} \rangle \quad (3)$$

[1 mark]

(BC4) At $x = L$, $y = 0$, therefore from (2):

$$0 = \frac{R_A L^3}{6} + \frac{M_A L^2}{2} - \frac{M_C \langle L - \frac{L}{4} \rangle^2}{2} \quad (4)$$

[1 mark]

Using simultaneous equations (3) and (4) to solve for R_A and M_A gives:

$$R_A = \frac{9M_C}{8L} \quad (5)$$

and:

$$M_A = \frac{3M_C}{16} \quad (6)$$

[2 marks]

From vertical equilibrium:

$$R_A + R_B = 0$$

Therefore, substituting (5) into this gives:

$$R_B = -\frac{9M_C}{8L}$$

[1 mark]

Taking moments about position B gives:

$$M_A + R_A L = M_B + M_C$$

Substituting (5) and (6) into this gives:

$$M_B = \frac{5M_C}{16}$$

[1 mark]

(b)

From (2), at $x = \frac{L}{4}$ (point C):

$$y = \frac{1}{EI} \left(\frac{R_A \left(\frac{L}{4}\right)^3}{6} + \frac{M_A \left(\frac{L}{4}\right)^2}{2} \right)$$

[1 mark]

Substituting (5) and (6) into this gives:

$$y = \frac{1}{EI} \left(\frac{9M_C L^2}{3072} + \frac{3M_C L^2}{512} \right) = \frac{1}{EI} \left(\frac{27M_C L^2}{3072} \right) \quad (7)$$

where,

$$M_C = 2 \left(F \times \frac{d}{2} \right) = Fd$$

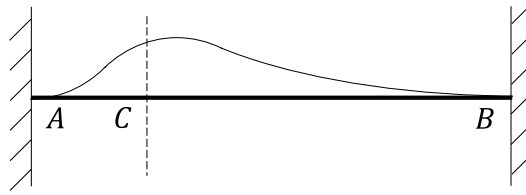
[2 marks]

Substituting this into (7) gives:

$$y = \frac{27FdL^2}{3072EI}$$

[2 marks]

(c)

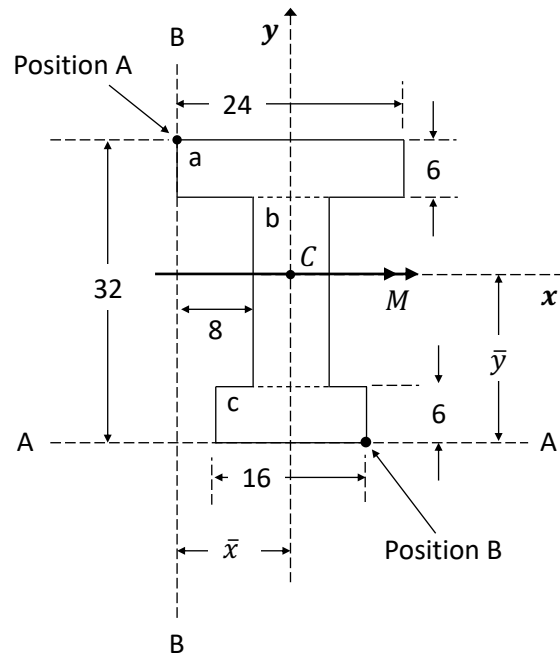


[5 marks]

2.

(a)

Position of Centroid, C



Total area,

$$A = (24 \times 6)_a + (8 \times 20)_b + (16 \times 6)_c = 400 \text{ mm}^2$$

[1 mark]

Taking moments about AA:

$$\bar{y} = \frac{(24 \times 6 \times 29)_a + (8 \times 20 \times 16)_b + (16 \times 6 \times 3)_c}{400} = 17.56 \text{ mm}$$

[2 marks]

Similarly, taking moments about BB:

$$\bar{x} = \frac{(6 \times 24 \times 12)_a + (20 \times 8 \times 12)_b + (6 \times 16 \times 12)_c}{400} = 12 \text{ mm}$$

[2 marks]

(b)

2nd Moments of Area and Product Moment of Area about the $x - y$ axes through C

Therefore, using the Parallel Axis Theorem,

$$\begin{aligned} I_{x'} &= (I_x + Ab^2)_a + (I_x + Ab^2)_b + (I_x + Ab^2)_c \\ &= \left(\frac{24 \times 6^3}{12} + 24 \times 6 \times (29 - 17.56)^2 \right) + \left(\frac{8 \times 20^3}{12} + 8 \times 20 \times (16 - 17.56)^2 \right) \\ &\quad + \left(\frac{16 \times 6^3}{12} + 16 \times 6 \times (3 - 17.56)^2 \right) \\ &= \mathbf{45,639.9 \text{ mm}^4} \end{aligned}$$

[2 marks]

and,

$$\begin{aligned} I_{y'} &= (I_y + Aa^2)_a + (I_y + Aa^2)_b + (I_y + Aa^2)_c \\ &= \left(\frac{6 \times 24^3}{12} + 6 \times 24 \times (12 - 12)^2 \right) + \left(\frac{20 \times 8^3}{12} + 20 \times 8 \times (12 - 12)^2 \right) \\ &\quad + \left(\frac{6 \times 16^3}{12} + 6 \times 16 \times (12 - 12)^2 \right) \\ &= 6912 + 853.33 + 2048 \\ &= \mathbf{9,813.33 \text{ mm}^4} \end{aligned}$$

[2 marks]

Also,

$$\begin{aligned} I_{x'y'} &= (I_{xy} + Aab)_a + (I_{xy} + Aab)_b + (I_{xy} + Aab)_c \\ &= (0 + 24 \times 6 \times (12 - 12) \times (29 - 17.65)) + (0 + 8 \times 20 \times (12 - 12) \times (16 - 17.56)) \\ &\quad + (0 + 16 \times 6 \times (12 - 12) \times (3 - 17.56)) \\ &= \mathbf{0 \text{ mm}^4} \end{aligned}$$

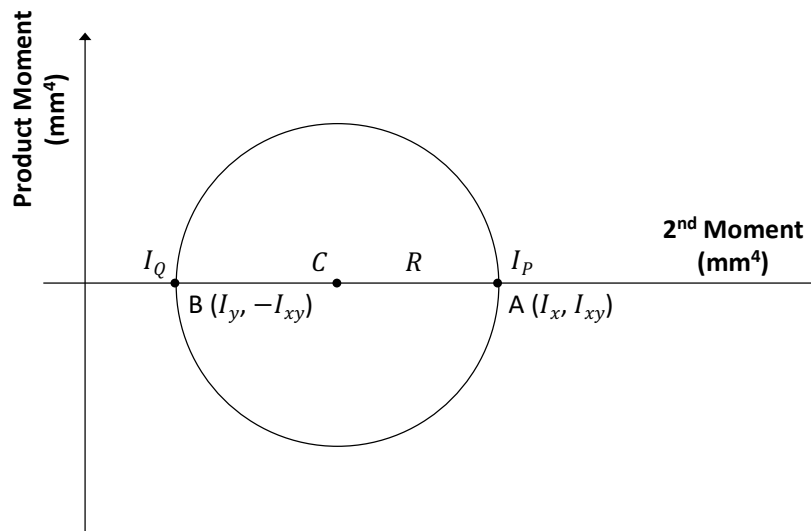
[1 mark]

(c)

Principal Second Moments of Area, direction of the Principal Axes and direction of the Neutral Axis

(By calculation)

Mohr's Circle



$$\text{Centre, } C = \frac{I_{x'} + I_{y'}}{2} = \frac{45,639.9 + 9,813.33}{2} = 27,726.17 \text{ mm}^4$$

$$\text{Radius, } R = \sqrt{\left(\frac{I_{x'} - I_{y'}}{2}\right)^2 + I_{x'y'}^2} = \sqrt{\left(\frac{45,639.9 - 9,813.33}{2}\right)^2 + 0^2} = 17,913.29 \text{ mm}^4$$

[3 marks]

Therefore, the Principal 2nd Moments of Area are:

$$I_P = C + R = 27,726.17 + 17,913.29 = 45,639.46 \text{ mm}^4$$

and,

$$I_Q = C - R = 27,726.17 - 17,913.29 = 9,812.88 \text{ mm}^4$$

[1 mark]

Directions of the Principal Axes

From the Mohr's circle above:

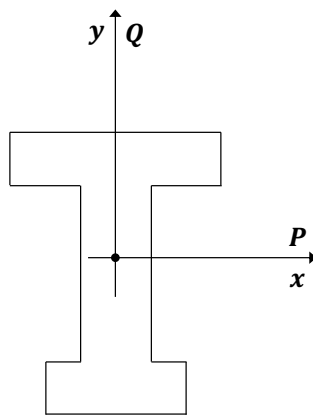
$$\sin 2\theta = \frac{I_{xy}}{R} = \frac{0}{17,913.29}$$

$$\therefore \theta = 0^\circ$$

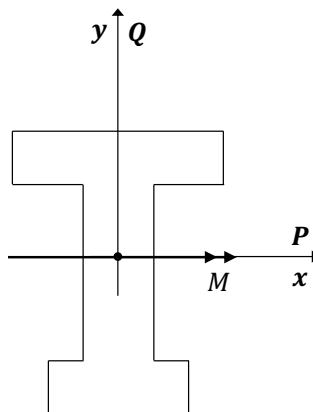
(this can also be seen from observation of the Mohr's Circle, in this case)

[1 mark]

Therefore, the Principal Axes are at 0° from the $x - y$ axes, as shown on the diagram below.



There is an applied Bending Moment, M , of 225Nm about the x -axis as shown below,



Resolve applied bending moment onto Principal Axes

As the applied Bending Moment is along the x -axis, and therefore the P -axis,

$$M_P = M = 325,000 \text{ Nmm}$$

and,

$$M_Q = 0 \text{ Nmm}$$

[1 mark]

Calculation of position of Neutral Axis

$$\sigma_b = \frac{M_P Q}{I_P} - \frac{M_Q P}{I_Q}$$

At the Neutral Axis, $\sigma_b = 0$, therefore,

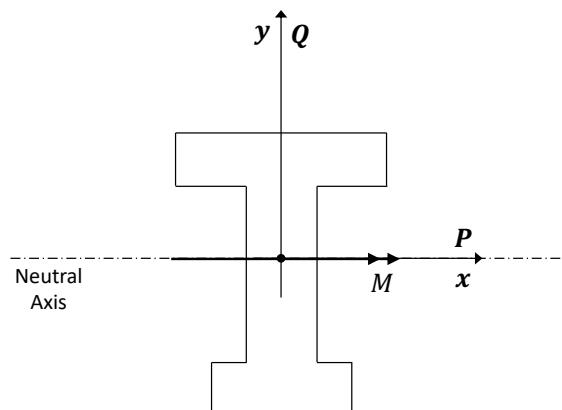
$$\begin{aligned} \frac{M_P Q}{I_P} - \frac{M_Q P}{I_Q} &= 0 \\ \therefore \frac{Q}{P} &= \frac{M_Q I_P}{M_P I_Q} \end{aligned}$$

Therefore, α , the angle between the neutral axis and the principal axes can be defined as,

$$\alpha = \tan^{-1} \left(\frac{Q}{P} \right) = \tan^{-1} \left(\frac{M_Q I_P}{M_P I_Q} \right) = \tan^{-1} \left(\frac{0 \times 45,639.46}{225,000 \times 9,812.88} \right) = 0^\circ$$

[2 marks]

Therefore, the neutral axis is at 0° from the principal axes as shown below,



The neutral axis is therefore at $(0^\circ - 0^\circ) = 0^\circ$ from the x -axis.

[2 marks]

(By observation)

As this section is symmetrical about the y -axis, it can be seen that,

$$I_P = I_{x'} = 45,639.46 \text{ mm}^4$$

and,

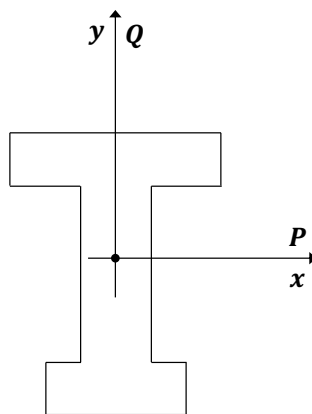
$$I_Q = I_{y'} = 9,812.88 \text{ mm}^4$$

[4 marks]

Also, since $I_{x'y'} = 0$

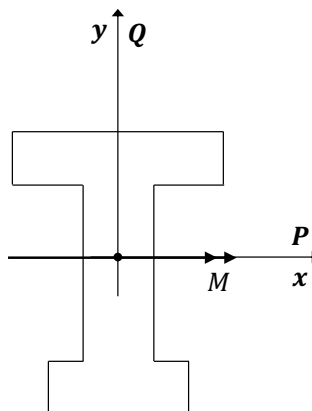
$$\theta = 0^\circ$$

Therefore, the Principal Axes are at 0° from the $x - y$ axes, as shown on the diagram below.



[1 mark]

There is an applied Bending Moment, M , of 325 Nm about the x -axis as shown below,



Resolve applied Bending Moment onto Principal Axes

As the applied Bending Moment is along the x -axis, and therefore the P -axis, it can be seen by observation that,

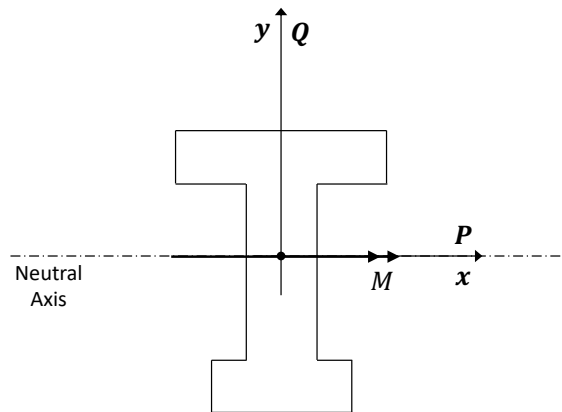
$$M_P = M = 325,000 \text{ Nmm}$$

and,

$$M_Q = 0 \text{ Nmm}$$

[1 mark]

Again, as the section is symmetrical about the y -axis, the angle between the $P - Q$ axis (and therefore also the $x - y$ axis) and the Neutral Axis, α , **must be at 0°** , as shown below,



The neutral axis is therefore at $(0^\circ - 0^\circ =) 0^\circ$ from the x -axis.

[4 marks]

(d)

Stresses at positions A and B

As above,

$$\sigma_b = \frac{M_P Q}{I_P} - \frac{M_Q P}{I_Q}$$

Therefore, the co-ordinates of point A on the $P - Q$ axes are required. As the $P - Q$ axis lies on the $x - y$ axis, the P co-ordinates are equal to the x co-ordinates and the Q co-ordinates are equal to the y co-ordinates.

[1 mark]

Therefore, for point A, $P = x = -12 \text{ mm}$ and $Q = y = 14.44 \text{ mm}$. Therefore, these $P - Q$ co-ordinates for position A can now be substituted into the equation for bending stress to give:

$$\sigma_{bA} = \frac{M_P Q}{I_P} - \frac{M_Q P}{I_Q} = \frac{325,000 \times 14.44}{45,639.46} - \frac{0 \times -12}{9,812.88}$$

$$\therefore \sigma_{bA} = \mathbf{102.83 \text{ MPa}}$$

[2 marks]

And for point B, $P = x = 12$ mm and $Q = y = -17.56$ mm. Therefore, these $P - Q$ co-ordinates for position A can now be substituted into the equation for bending stress to give:

$$\sigma_{bB} = \frac{M_P Q}{I_P} - \frac{M_Q P}{I_Q} = \frac{325,000 \times -17.56}{45,639.46} - \frac{0 \times 12}{9,812.88}$$

$$\therefore \sigma_{bB} = -125.05 \text{ MPa}$$

[2 marks]

3.

(a)

$$\sigma_z = \frac{R_i^2 p_i - R_o^2 p_o}{(r_o^2 - r_i^2)}$$

[2 marks]

Substituting in values

$$\sigma_z = \frac{50^2 \times 140 - 100^2 \times 20}{(100^2 - 50^2)} = 20 \text{ MPa}$$

[3 marks]

(b)

Lame's Equations

$$\sigma_r = A - \frac{B}{r^2}$$

$$\sigma_\theta = A + \frac{B}{r^2}$$

[2 marks]

Using radial equation and applying BCs

At ID:

$$\sigma_r = -140 = A - \frac{B}{50^2}$$

At OD:

$$\sigma_r = -20 = A - \frac{B}{100^2}$$

[2 marks]

Subtract ID from OD

$$-20 + 140 = 120 = -\frac{B}{100^2} + \frac{B}{50^2}$$

$$\therefore 120 = B \left(\frac{1}{2500} - \frac{1}{10000} \right)$$

$$\therefore B = \frac{120}{0.0003} = 400000$$

[3 marks]

Therefore, from OD:

$$A = -20 + \frac{400000}{10000} = 20$$

[2 marks]

Calculating the hoop stress value at the ID:

$$\sigma_{\theta} = 20 + \frac{400000}{50^2} = \mathbf{180 \text{ MPa}}$$

[3 marks]

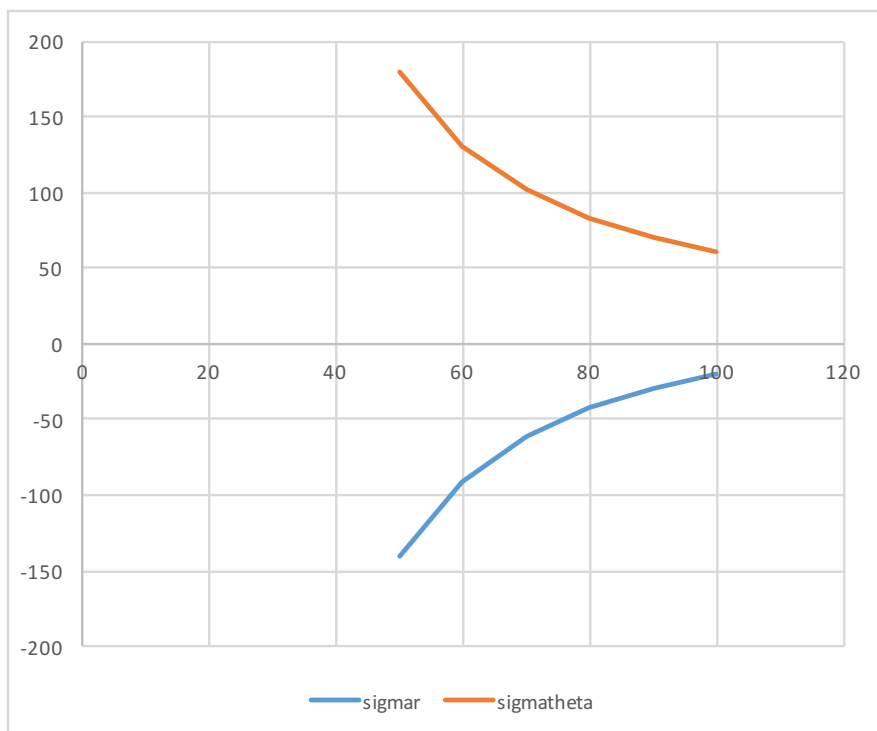
and at the OD:

$$\sigma_{\theta} = 20 + \frac{400000}{100^2} = \mathbf{60 \text{ MPa}}$$

[3 marks]

(c)

Distributions of hoop and radial stress with r



[5 marks]

4.

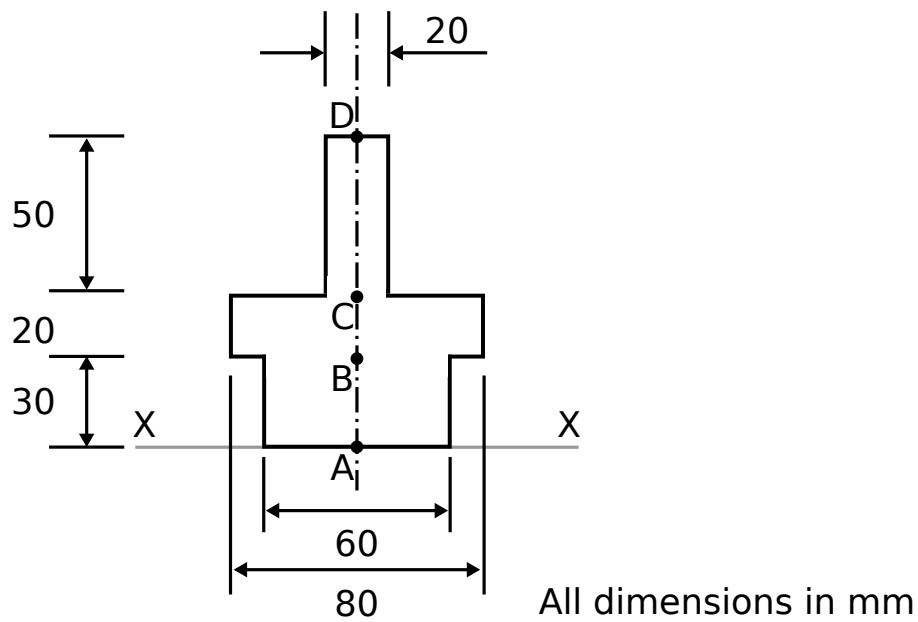
The section shown in Figure Q4 carries a shear force, $S = 45 \text{ kN}$ down the vertical centre line.

(a)

Using first moment of area:

$$\bar{y} = \frac{\sum A\bar{y}_n}{A_T}$$

[1 mark]



	Width	Length	\bar{y}_n	A	$A\bar{y}_n$
A	20	50	75	1000	75000
B	80	20	40	1600	64000
C	60	30	15	1800	27000
			Sum:	4400	166000
				\bar{y}	37.7

[3 marks]

(b)

$$I = \sum \left(\frac{BD^3}{12} + A(\bar{y}_n - \bar{y})^2 \right)$$

[2 marks]

$\bar{y}_n - \bar{y}$	$A(\bar{y}_n - \bar{y})^2$	$\frac{BD^3}{12}$	$A(\bar{y}_n - \bar{y})^2 + \frac{BD^3}{12}$
37.3	1389256.198	208333.3333	1597589.532
2.3	8264.46281	53333.33333	61597.79614
-22.7	929752.0661	135000	1064752.066
		<i>I</i>	2723939.4

[3 marks]

(c)

$$\tau = \frac{SQ}{Iz}$$

[1 mark]

A & D are free surfaces so:

$$\tau_A = \tau_D = 0$$

[1 mark]

Can use area below at B, 2 values due to section change:

$$\tau_{B1} = \frac{45000 \times (60 \times 30) \times (37.7 - 15)}{2723939.4 \times 60} = \mathbf{11.3 \text{ MPa}}$$

[2 marks]

$$\tau_{B2} = \frac{45000 \times (60 \times 30) \times (37.7 - 15)}{2723939.4 \times 80} = \mathbf{8.4 \text{ MPa}}$$

[2 marks]

At G (centroid), need to consider the effect of two areas:

In this case $Q = \sum A(y_n - y)$

$$Q = ((20 \times 50) \times (75 - 37.7)) + \left(80 \times (50 - 37.7) \times \left(\frac{50 - 37.7}{2} \right) \right)$$

$$Q = 37272.8 + 6051.6 = 43324.4$$

and therefore:

$$\tau_G = \frac{45000 \times 43324.4}{2723939.4 \times 80} = \mathbf{8.9 \text{ MPa}}$$

[2 marks]

Two values of shear stress at C due to section change (we can use area above):

$$\tau_{C1} = \frac{45000 \times (20 \times 50) \times (75 - 37.7)}{2723939.4 \times 80} = \mathbf{7.7 \text{ MPa}}$$

[2 marks]

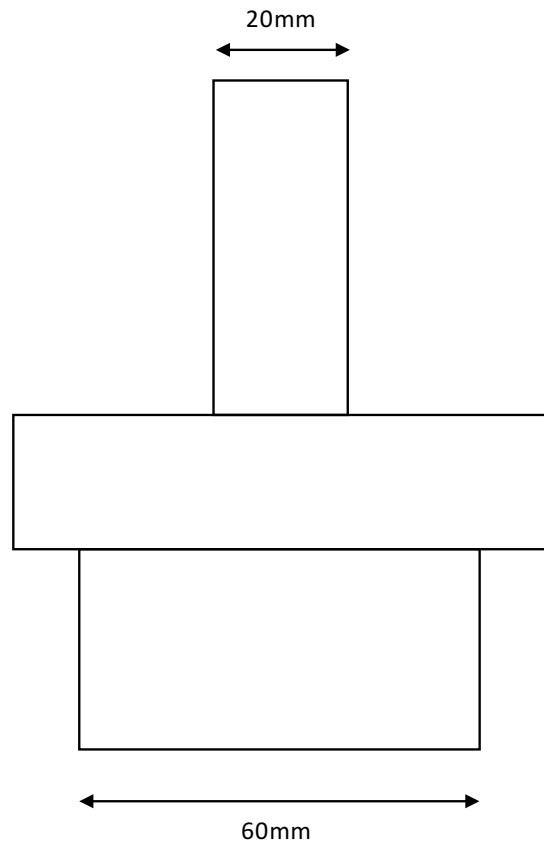
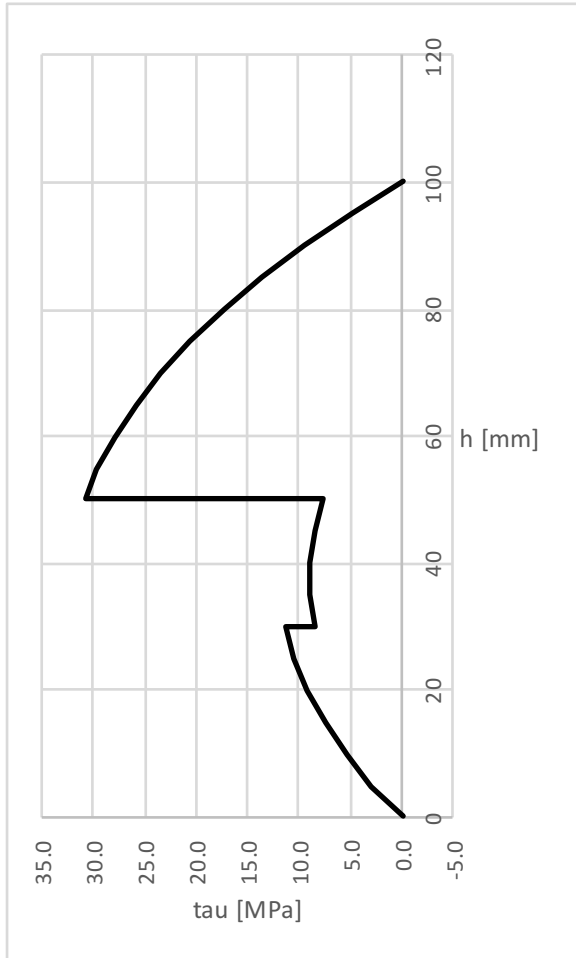
$$\tau_{C2} = \frac{45000 \times (20 \times 50) \times (75 - 37.7)}{2723939.4 \times 20} = \mathbf{30.8 \text{ MPa}}$$

[2 marks]

(d)

h	τ	
0	0.0	A
5	2.9	
10	5.4	
15	7.5	
20	9.2	
25	10.4	
30	11.3	B1
30	8.4	B1
35	8.9	
37.7	8.9	G
40	8.9	
45	8.5	
50	7.7	C1
50	30.8	C2
55	29.6	
60	27.9	
65	25.9	
70	23.4	
75	20.6	
80	17.3	
85	13.6	
90	9.5	
95	4.9	
100	0.0	D

[2 marks]

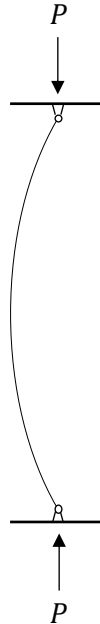


[2 marks]

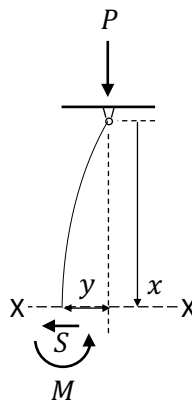
5.

(a)

Below is a diagrammatic representation of a pinned-pinned strut.



Sectioning this beam in order to determine the bending moment:



[1 mark]

Taking moments about the section position, X-X:

$$M = Py \quad (1)$$

2nd order differential equation for a beam under bending:

$$EI \frac{d^2y}{dx^2} = M$$

Substituting (1) into this:

$$EI \frac{d^2 y}{dx^2} = Py$$

[1 mark]

Let $y = A_0 e^{\alpha x}$:

$$\therefore EI\alpha^2 + P = 0$$

and:

$$\alpha = \pm \sqrt{\frac{P}{EI}} i$$

We get:

$$y = A \sin\left(\sqrt{\frac{P}{EI}} x\right) + B \cos\left(\sqrt{\frac{P}{EI}} x\right) \quad (2)$$

where A_0 , A and B are constants.

[1 mark]

Boundary conditions:

(BC1) At $x = 0$, $y = 0$, therefore from (2):

$$B = 0$$

[1 mark]

(BC2) At $x = L$, $y = 0$, therefore from (2):

$$A \sin\left(\sqrt{\frac{P}{EI}} L\right) = 0$$

Since $A \neq 0$ for non-trivial solution:

$$\sqrt{\frac{P}{EI}} L = n\pi$$

[1 mark]

$$\therefore P = \frac{n^2 \pi^2 EI}{L^2} \quad (3)$$

where $n = 1, 2, \dots$

[2 marks]

(b)

$$\sigma = \frac{P}{A}$$

[2 marks]

Substituting the buckling load (from (3)) into this equation:

$$\sigma = \frac{n^2 \pi^2 EI}{L^2 A} = \frac{n^2 \pi^2 E A k^2}{L^2 A} = \frac{\pi^2 E}{\left(\frac{L}{k}\right)^2}$$

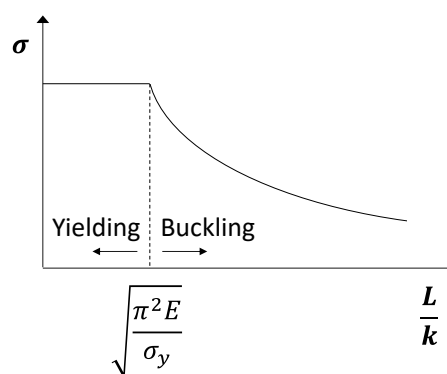
[2 marks]

Therefore when $\sigma = \sigma_y$:

$$\frac{L}{k} = \sqrt{\frac{\pi^2 E}{\sigma_y}}$$

[2 marks]

The figure below therefore displays the transition between yielding and buckling in terms of slenderness ratio.



[4 marks]

(c)

$$\frac{L}{k} = \sqrt{\frac{\pi^2 E}{\sigma_y}} = \sqrt{\frac{\pi^2 \times 70,000}{230}} = 54.81 \quad (4)$$

[2 marks]

where k is calculated by:

$$I = \frac{bd^3}{12} = \frac{t^4}{12} = t^2 k^2$$

[2 marks]

where $b = d = t$ and $I = Ak^2 = bdk = t^2 k^2$.

$$\therefore k = 8.66 \text{ mm}$$

[2 marks]

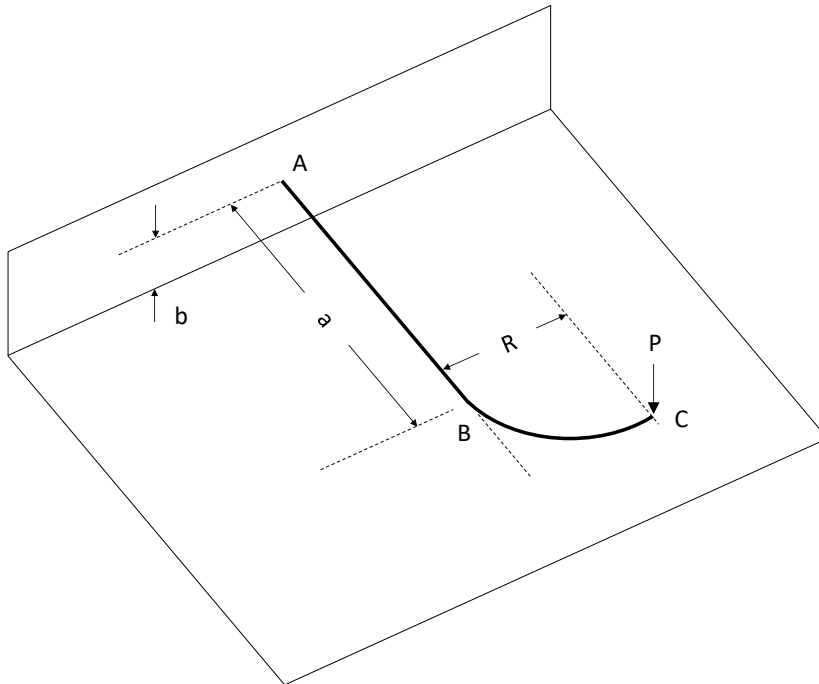
Therefore from (4):

$$\mathbf{L = 474.66 \text{ mm}}$$

[2 marks]

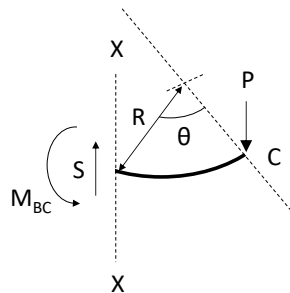
6.

Label bracket dimensions:



Section BC (*bending only*)

Free Body Diagram:



[2 marks]

Taking moments about X-X:

$$M_{AB} = PR\sin\theta$$

[2 marks]

Substituting this into the equation for Strain Energy in a beam under bending gives,

$$U_{BC} = \int_0^x \frac{M_{AB}^2}{2EI} dx = \int_0^{\pi/2} \frac{(PR\sin\theta)^2}{2EI} R d\theta$$

where,

$$dx = R d\theta$$

$$\therefore U_{BC} = \frac{P^2 R^3}{2EI} \int_0^{\pi/2} \sin^2 \theta d\theta \quad (1)$$

[2 marks]

Trigonometric Identities:

$$\sin^2 \theta + \cos^2 \theta = 1 \quad (2)$$

and,

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta \quad (3)$$

Rearranging equation (2) gives,

$$\cos^2 \theta = 1 - \sin^2 \theta$$

Substituting this into equation (3) gives,

$$\cos 2\theta = 1 - 2\sin^2 \theta$$

$$\therefore \sin^2 \theta = \frac{1 - \cos 2\theta}{2}$$

[2 marks]

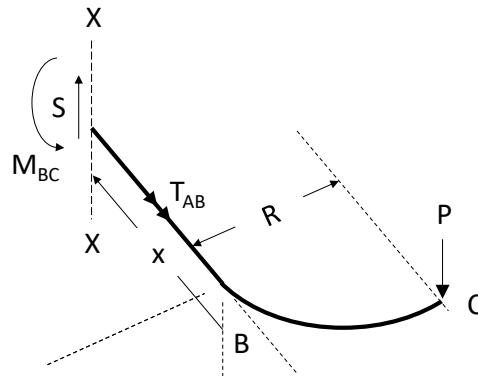
Substituting this into equation (1) gives,

$$\begin{aligned} U_{BC} &= \frac{P^2 R^3}{2EI} \int_0^{\pi/2} \left(\frac{1 - \cos 2\theta}{2} \right) d\theta \\ &= \frac{P^2 R^3}{2EI} \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4} \right]_0^{\pi/2} = \frac{P^2 R^3}{2EI} \left(\left(\frac{\pi}{4} - \frac{\sin \pi}{4} \right) - \left(\frac{0}{2} - \frac{\sin(0)}{4} \right) \right) \\ \therefore U_{BC} &= \frac{P^2 R^3 \pi}{8EI} \end{aligned}$$

[2 marks]

Section AB

Free Body Diagram:



[2 marks]

Bending

Taking moments about X-X:

$$M_{AB} = P(R + x)$$

[2 marks]

Substituting this into the equation for Strain Energy in a beam under bending gives,

$$\begin{aligned} U_{AB}^{bending} &= \int_0^a \frac{M_{AB}^2}{2EI} dx = \int_0^a \frac{(P(R+x))^2}{2EI} dx = \frac{P^2}{2EI} \int_0^a (R+x)^2 dx = \frac{P^2}{2EI} \int_0^a (R^2 + 2Rx + x^2) dx \\ &= \frac{P^2}{2EI} \left[R^2x + Rx^2 + \frac{x^3}{3} \right]_0^a \\ \therefore U_{AB}^{bending} &= \frac{P^2}{2EI} \left(R^2a + Ra^2 + \frac{a^3}{3} \right) \end{aligned}$$

[2 marks]

Torsion

Taking moments about X-X:

$$T_{AB} = PR$$

[2 marks]

Substituting this into the equation for Strain Energy in a beam under bending gives,

$$U_{AB}^{torsion} = \int_0^a \frac{T_{AB}^2}{2GJ} dx = \int_0^a \frac{(PR)^2}{2GJ} dx = \frac{P^2R^2}{2GJ} \int_0^a 1 dx = \frac{P^2R^2}{2GJ} [x]_0^a = \frac{P^2R^2}{2GJ} (a - 0)$$

$$\therefore U_{AB}^{torsion} = \frac{P^2 R^2 a}{2GJ}$$

[2 marks]

$$U_{AB} = U_{AB}^{bending} + U_{AB}^{torsion} = \frac{P^2}{2EI} \left(R^2 a + Ra^2 + \frac{a^3}{3} \right) + \frac{P^2 R^2 a}{2GJ}$$

[1 mark]

Total Strain Energy

$$U = U_{AB} + U_{BC} = \frac{P^2}{2EI} \left(R^2 a + Ra^2 + \frac{a^3}{3} \right) + \frac{P^2 R^2 a}{2GJ} + \frac{P^2 R^3 \pi}{8EI}$$

$$\therefore U = \frac{P^2}{2} \left(\frac{R^2 a}{EI} + \frac{Ra^2}{EI} + \frac{a^3}{3EI} + \frac{R^2 a}{GJ} + \frac{R^3 \pi}{4EI} \right)$$

[1 mark]

Deflection Calculation using Castigliano's Theorem

$$u_{vA} = \frac{\delta U}{\delta P} = P \left(\frac{R^2 a}{EI} + \frac{Ra^2}{EI} + \frac{a^3}{3EI} + \frac{R^2 a}{GJ} + \frac{R^3 \pi}{4EI} \right)$$

[1 mark]

where,

$$I = \frac{\pi d^4}{64} = \frac{\pi \times 28^4}{64} = 30171.86 \text{ mm}^4$$

and,

$$J = \frac{\pi d^4}{32} (= 2I) = 60343.71 \text{ mm}^4$$

$$\therefore u_{vA} = 500 \left(\frac{500^2 \times 1500}{210000 \times 30171.86} + \frac{500 \times 1500^2}{210000 \times 30171.86} + \frac{1500^3}{3 \times 210000 \times 3.02 \times 10^{-8}} + \frac{500^2 \times 1500}{77000 \times 6.03 \times 10^{-8}} \right. \\ \left. + \frac{500^3 \times \pi}{4 \times 210000 \times 3.02 \times 10^{-8}} \right)$$

$$= 257.5 \text{ mm}$$

Therefore, as the calculated deflection at point C is more than 250 mm, **point C will make contact with the ground.**

[2 marks]